THE ROLE OF SURFACE VARIABLES IN THE VACUUM STRUCTURE OF YANG-MILLS THEORY*

S. WADIA and T. YONEYA[1]

Department of Physics, The City College of the City University of New York, New York, N.Y. 10031, USA

Received 17 December 1976

The structure of the vacuum in the SU(2) Yang-Mills theory is discussed for general gauges. The original discussions given by Callan et al. and Jackiw and Rebbi are restricted to a particular class of gauge conditions. We show that the periodic vacua and the "vacuum seizing" in the presence of massless fermions can be realized in any gauge by recognizing the independent dynamical role of surface variables defined at spatial infinity.

The relevance of the Euclidean pseudo-particles [1] for the vacuum structure of the Yang-Mills theory, in weak coupling, was recently discussed by Callan et al. and Jackiw and Rebbi [2]. They found a series of degenerate vacua $|n\rangle$ (n = integer) which are classified by the gauge topology; the pseudo-particle solution describes tunneling between these in the WKBJ approximation [3]. The true vacuum is a linear superposition of $\{|n\rangle\}$, $|\theta\rangle = \Sigma_n e^{in\theta}|n\rangle$, $0 \leqslant \theta \leqslant 2\pi$. The different values of the angle θ (a measure of spontaneous P and T violation) defines different theories which do not transform into each other. In the presence of massless fermions, although tunneling is suppressed, the vacuum must still be $|\theta\rangle$ by the requirement of cluster decomposition. The U(1) chiral symmetry is then spontaneously violated without a Nambu–Goldstone boson as previously suggested by Kogut and Susskind [4]. These qualitative features are reminiscent of the Schwinger model [4].

However, it seems that previous arguments are only valid within a very restricted class of gauge conditions. Callan et al. used the $A_0 = 0$ gauge. Although Jackiw and Rebbi did not specify the gauge, it is evident that their discussion does not apply, for example, to the axial gauge [5]. It is therefore desirable to clarify the situation and to extend the above picture to any gauge. In this letter we establish the validity of the periodic vacuum structure and the associated vacuum seizing in a wider class of gauge conditions by recognizing the dynamic role played by boundary conditions in terms of surface variables [6].

By now it is well known that a WKBJ description [3] of tunneling can be given by the Euclidean classical solution which interpolates between the initial and final states. Therefore, we restrict ourselves to EFT. We have the initial and final condition $F_{\mu\nu}(\boldsymbol{x}, x_4 = \pm\infty) \approx 0$. This implies $A_\mu(\boldsymbol{x}, x_4 = \pm\infty) \approx \omega^{-1}\partial_\mu\omega$. Since we are in the vacuum sector we impose the boundary condition

$$A_\mu(r = \infty, x_4) \approx \omega^{-1}\partial_\mu\omega, \qquad \omega = \exp(i\,\tfrac{1}{2}\tau_a\theta_a). \tag{1}$$

The method of handling the boundary condition (1) in terms of an action principle was described in ref. [6]. The action is

$$S = -\frac{1}{g^2}\int d^4x\, \mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) - \frac{1}{g^2}\int dx_4 \int d^2\sigma_i\, \mathrm{Tr}\, g_{\mu i}(A_\mu - \omega^{-1}\partial_\mu\omega), \tag{2}$$

which is invariant under the gauge transformation

$$A_\mu \to U^{-1}A_\mu U + U^{-1}\partial_\mu U, \qquad \omega \to \omega U, \qquad g_{\mu\nu} \to U^{-1}g_{\mu\nu}U. \tag{3}$$

* Work supported by National Science Foundation under Grant No. PHY75-07376 A01 and in part by the Research Foundation of the City University of New York under Grant No. 11118.

[1] On leave from Department of Physics, Hokkaido University, Sapporo, Japan.

It is important to recognize the $\theta^a(x)$ and $g_{\mu\nu}(x)$, (which are defined only at spatial infinity) as independent dynamical variables besides the original field A_μ, because there are gauges in which the surface value of A_μ does *not* vanish for any finite time and changes in time. As we shall see later, this is crucial to realize the vacuum structure in general gauges. The variational principle then implies the equations of motion and the boundary condition (1). The quantum states are functionals of A_μ and θ^a and must satisfy the constraint (1).

The existence of multiple classical vacua stems from the nonvanishing of the Pontrjagin class number [1]

$$q = -(1/32\pi^2)\,\epsilon_{\mu\nu\alpha\beta} \int d^4x\, \mathrm{Tr}(F_{\mu\nu}F_{\alpha\beta}), \tag{4}$$

in Euclidean space. If the fields are not singular we have the identity [1]

$$(1/16\pi^2)\,\epsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}(F_{\mu\nu}F_{\alpha\beta}) = \partial_\mu J^A_\mu, \qquad J^A_\mu = (1/4\pi^2)\,\epsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}(A_\nu\partial_\alpha A_\beta + \tfrac{2}{3}A_\nu A_\alpha A_\beta). \tag{5, 6}$$

Thus, q can always be written as a four dimensional surface integral. In particular, in the $A_0 = 0$ gauge the space infinity does not contribute and we have [2]

$$q = n_+ - n_-, \tag{7}$$

where

$$n_\pm = -(1/24\pi^2) \int d\boldsymbol{x}\, \epsilon_{ijk}\, \mathrm{Tr}(A_iA_jA_k)\Big|_{x_4=\pm\infty}. \tag{8}$$

Since q is an integer, the existence of degenerate multiple vacua with different "quantum number" $n_\pm$ is quite easily visualized as discussed by CDG and JR [2]. However, in the axial gauge [5], for example, it is evident that (8) has no meaning because $A_i \to 0$ for x_4, while $q \neq 0$ due to the contribution from spatial infinity.

Now, we shall show that in any gauge the winding number q can be casted into the form (7), if we include the additional contribution coming from spatial infinity. To do so, it is convenient to introduce a parametrization for $\omega(x)$ ‡

$$\omega(x) = u_0 + iu_k\tau_k, \quad u_0^2 + u_ku_k = 1, \quad u_0 = \cos\psi(x), \quad u_1 = \sin\psi(x)\sin\theta(x)\cos\phi(x),$$
$$u_2 = \sin\psi(x)\sin\theta(x)\sin\phi(x), \quad u_3 = \sin\psi(x)\cos\theta(x), \quad 0 \leqslant \psi \leqslant \pi, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi \leqslant 2\pi. \tag{9}$$

Then

$$q = -\frac{1}{2}\left[\int d\boldsymbol{x}\, J^A_0\right]_{x_4=-\infty}^{x_4=+\infty} - \frac{1}{2}\int dx_4 \int d^2\sigma_i J^A_i. \tag{10}$$

Now

$$-\tfrac{1}{2}\int d^2\sigma_i J^A_i = (1/24\pi^2)\int d^2\sigma_i\, \epsilon_{i\mu\nu\lambda}\, \mathrm{Tr}(\omega^{-1}\partial_\mu\omega\cdot\omega^{-1}\partial_\nu\omega\cdot\omega^{-1}\partial_\lambda\omega) = (1/24\pi^2)\int d^2\sigma_i\, \partial_\mu S^i_\mu \tag{11}$$

where

$$S^i_\mu = \tfrac{1}{12}\,\epsilon_{i\mu\nu\lambda}\,(\tfrac{1}{2}\psi - \tfrac{1}{4}\sin 2\psi)\,\partial_\nu\cos\theta\,\partial_\lambda\phi.$$

Hence

$$q = n_+ - n_-$$

where

‡ The surface variables are defined at $r = \infty$ for all times. Hence the limit $x_4 \to \pm\infty$ is implicitly taken after the limit $r \to \infty$.

$$n_\pm = \left[(1/24\pi^2)\int \mathrm{d}\boldsymbol{x}\,\epsilon_{ijk}\,\mathrm{Tr}(A_iA_jA_k) + (1/2\pi^2)\int \mathrm{d}^2\sigma_i\,S_i^0\right]_{x_4=\pm\infty} \tag{12}$$

(provided ψ is a single valued function of spatial co-ordinates). We note that even if $A_i = 0$ at $x_4 = \pm\infty$ and thus the first term in (11) vanishes, the second term in general does *not* vanish. This inevitably requires us to treat the surface variables as independent dynamical variables.

We are thus led to the following classification of gauge conditions:

(a) Non zero contribution to $n_\pm$ only from the volume integral.

(b) Non zero contribution to $n_\pm$ only from the surface integral.

(c) Non zero contribution to $n_\pm$ from both volume and surface integrals.

The $A_0 = 0$ gauge conforms to case (a). The axial gauge $A_3 = 0$ and the radial gauge $\hat{x}_iA_i = 0$ conform to case (b). Later we shall indicate that the Coulomb gauge also conforms to case (a).

In order to realize the importance of including the surface term in the case (b), we explicitly discuss the BPST solution [1] in the radial and axial gauges. As an interesting example, we also consider the Coulomb gauge. For the case of $A_0 = 0$ gauge see Gervais and Sakita [7]. The gauge transformed field is

$$A_\mu^U = U^{-1}A_\mu^P U + U^{-1}\partial_\mu U, \qquad A_\mu^P = \frac{x^2}{x^2+\lambda^2}\left[\frac{x_4 - \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{x}}{\sqrt{x^2}}\right]\partial_\mu\left[\frac{x_4 + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{x}}{\sqrt{x^2}}\right]. \tag{13, 14}$$

Since $A_\mu^P \to O(1/r)$ as $r\to\infty$ we have $A_\mu^U \to U^{-1}\partial_\mu U$ as $r\to\infty$. Hence $\lim_{r\to\infty} U(r, x_4) = \omega$.

Radial gauge: $\hat{x}_iA_i = 0$. Due to spherical symmetry we may set $U = \exp\{\mathrm{i}\hat{x}^a\,\tfrac{1}{2}\tau^a f(r, x_4)\}$. We require $f(r=0, x_4) = 0(\mathrm{mod}\ 2\pi)$, since U be regular at $r = 0$. Otherwise, we will induce a singularity in the gauge field and (5) is inapplicable.

$$f(r, x_4) = \frac{-2x_4}{\sqrt{x_4^2+\lambda^2}}\tan^{-1}\left(\frac{r}{\sqrt{x_4^2+\lambda^2}}\right), \tag{15}$$

then using $\psi = \lim_{r\to\infty}\tfrac{1}{2}f(r, x_4)$ we have $\psi = \mp\tfrac{1}{2}\pi$ at $x = \pm\infty$.

Hence

$$(1/2\pi^2)\int S_0^i\,\mathrm{d}^2\sigma_i\Big|_{x_4=\pm\infty} = \pm(1/8\pi)\int \mathrm{d}(\cos\theta)\,\mathrm{d}\phi = \pm\tfrac{1}{2},$$

and we have $n_+ - n_- = +1$. From this we may conclude that the "classical" degenerate vacua are "eigenstates" of $(1/2\pi^2)\int S_0^i\,\mathrm{d}^2\sigma_i$ with eigenvalues $n - 1/2$, (n = integer). Clearly the adjacent classical vacua are connected by the transformation $T = \exp(2\pi\,\mathrm{i}\int P^a\hat{x}^a\,\mathrm{d}w)$, where P^a is the momentum conjugate to θ^a.

Axial gauge [5]. $A_3 = 0$ with boundary condition $\lim_{x_3\to+\infty} A_0 = -\lim_{x_3\to-\infty} A_0$. Then

$$U = \exp\left(\frac{2\mathrm{i}}{\sqrt{x_1^2+x_2^2+x_4^2+\lambda^2}}\right)\tan^{-1}\left(\frac{x_3}{\sqrt{x_1^2+x_2^2+x_4^2+\lambda^2}}\right)\left(\frac{x_2\tau_1}{2} - \frac{x_1\tau_2}{2} + \frac{x_4\tau_3}{2}\right). \tag{17}$$

In this gauge it is convenient to use

$$S_\mu^i = -\epsilon_{i\mu\nu\lambda}\cos\theta\,\partial_\nu(\tfrac{1}{2}\psi - \tfrac{1}{4}\sin 2\psi)\,\partial_\lambda\phi, \tag{18}$$

which has the same divergence as the previous expression (12). The surface integral is performed over a large rectangular box. The integral over the planes $x_1 = \pm\infty$ and $x_2 = \pm\infty$ gives no contribution because $\phi = \phi(x_1, x_2)$ and $\partial_3\psi$ becomes zero.

Hence we obtain

$$(1/2\pi^2)\int d^2\sigma_i S_0^i = \left[(1/2\pi^2)\int d\sigma_3\, S_0^3\right]_{x_3=-\infty}^{x_3=+\infty}$$
$$= \left[(1/2\pi^2)\int \cos\theta\,(\partial_1(\tfrac{1}{2}\psi - \tfrac{1}{4}\sin 2\psi)\partial_2\phi + \partial_2(-\tfrac{1}{2}\psi + \tfrac{1}{4}\sin 2\psi)\partial_1\phi\, dx_1\, dx_2\right]_{x_3=-\infty}^{x_3=+\infty} \tag{19}$$

Observing $\cos\theta \to \pm 1$ as $x_4 \to \pm\infty$ and $\psi \to \pm\frac{1}{2}$ as $x_3 \to \pm\infty$, we have

$$\left[(1/2\pi^2)\int d^2\sigma_i S_0^i\right]_{x_4=\pm\infty} = (1/2\pi^2)\left[\oint \psi\left(\frac{\partial\phi}{\partial x_1}dx_1 + \frac{\partial\phi}{\partial x_2}dx_2\right)\right]_{x_3=+\infty} = \pm\tfrac{1}{2}. \tag{20}$$

Here again we have $n_+ - n_- = +1$. In this case, the transformation that connects the adjacent classical vacua is $T = \exp(2\pi i P_3)$ where

$$P_3 = \frac{1}{2}\int [p^3(x_3 = +\infty) + p^3(x_3 = -\infty)]\, d\sigma_3$$

Coulomb gauge: $\partial_i A_i = 0$. By setting $U = \exp(i\frac{1}{2}\tau^a\hat{x}^a f(r, x_4)), f(0, x_4) = 0 \pmod{2\pi}$ we have

$$\frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} = \frac{2}{r^2}\left\{\frac{2rx_4}{r^2 + x_4^2 + \lambda^2}\cos f - \frac{r^2 - x_4^2 - \lambda^2}{r^2 + x_4^2 + \lambda^2}\sin f\right\} - \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\frac{2x_4}{r^2 + x_4^2 + \lambda^2}. \tag{21}$$

Although it is difficult to solve this equation exactly, we can easily see that (21) implies $\lim_{r\to\infty} \partial f/\partial x_4(r, x_4) = 0(1/r)$ which leads to the neglect of the surface term. Therefore the Coulomb gauge conforms to case (a). The existence of degenerate multiple vacua can be seen by considering (21) at $x_4 = \pm\infty$. It has infinitely many distinct solutions $f_n(r)$ satisfying $f_n(0) = 0$ and $f_n(\infty) = 2n\pi$. To see this we put $\ln r = s$, then at $x_4 = \pm\infty$ (21) becomes

$$\frac{d^2f}{ds^2} = -\frac{df}{ds} + 2\sin f. \tag{22}$$

If s is regarded as "time" this is nothing but the equation of motion of a point particle in a periodic potential $2\cos f$ with frictional force $-(df/ds)$. For the sufficiently distant "past", the solution of (22) satisfying $f(s = -\infty) = 0$ is $f(s) = ce^s$ where c is an arbitrary constant. It is qualitatively obvious that we can always arrange the constant c such that $f(s = \infty) = 2n\pi$. $f_n(r)$ defines a mapping of winding number n from $S^3(= R^3 + \infty)$ to S^3 (= group manifold). Thus, even after we impose the gauge condition $\partial_i A_i^a = 0$ and boundary condition $U(0) = \pm 1$ and $U(\infty) = \pm 1$ there remains the freedom of discrete time-independent gauge transformation, hence the discussions of CDG and JR apply also to this case.

In the presence of massless fermion, the conserved but gauge variant axial current [2] is

$$J_\mu^5 = i\bar{\psi}\gamma_\mu\gamma_5\psi - J_\mu^A. \tag{23}$$

Although $\partial_\mu J_\mu^5 = 0, (d/dt)Q^5 \neq 0$ if one cannot neglect the surface term. ($Q^5 = \int d^3x\, J_0^5$). However, because of the existence of surface variables as independent dynamical variables we can define a new conserved axial charge

$$\hat{Q}^5 = Q^5 - (1/2\pi^2)\int d^2\sigma_i\, S_0^i + \tfrac{1}{2}.$$

By (11) $(d/dt)\hat{Q}^5 = 0$ for any gauge. Also $-\hat{Q}^5|n\rangle = 2n|n\rangle$.

This implies [2] that vacuum tunneling is forbidden in the presence of massless fermions. However, by the requirement of cluster decomposition [2, 4] the true vacuum is $|\theta\rangle = \Sigma_n\, e^{in\theta}|n\rangle$. Thus, in any gauge the "vacuum seizes" [4] with no Nambu-Goldstone boson corresponding to the spontaneous violation of U(1) chiral symmetry.

Conclusion. Vacuum periodicity is not an artifact of the $A_0 = 0$ gauge (or of the class of gauges we have referred to as (a)).

In other gauges the additional dynamical variables on the surface at spatial infinity play an essential role in describing the vacuum structure of Yang-Mills field.

It is a pleasure to acknowledge Professor Bunji Sakita for encouragement, stimulating discussions and critical comments.

References

[1] A.M. Polyakov, Phys. Lett. 59B (1975) 82;
Belavin, Polyakov, Schwartz, Tyupkin, Phys. Lett. 59B (1975) 85.
[2] C.G. Callan, R.F. Dashen and D.J. Gross, Phys Lett. 63B (1976) 334,
R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37 (1976) 172,
That pseudo particle solutions may have relevance for quantum mechanical tunneling from one vacuum to another, which is gauge rotated, was suggested in G 't Hooft, Phys. Rev. Lett. 37 (1976) 8.
[3] D. McLaughlin, J. Math. Phys. 13 (1972) 1099;
K. Freed and J. Chew, Phys. 56 (1972) 692.
[4] J. Kogut and L. Susking, Phys. Rev. D11 (1975) 3594,
J Lowenstein and A. Swieca, Ann Phys. 68 (1971) 172.
[5] C. Bernard and E. Weinberg, Columbia Univ., preprint C-2271-87
[6] J.L. Gervais, B. Sakita and S. Wadia, Phys. Lett. 63B (1976) 55,
T. Regge and C Teitelboim, Ann. of Phys. 88 (1974) 286.
[7] J.L. Gervais and B. Sakita, CCNY preprint-HEP-76/11.